The Oranges Problem

A man goes to a fruit stand and buys half of the oranges and half of an orange. Then, another man goes to the same fruit stand and buys half of the oranges that are left and half of an orange. Finally, a third man goes to the same fruit stand and buys half of the oranges that are left and half of an orange. After that, there are no oranges left. How many oranges were there at the fruit stand before the first man bought any?

For a purpose that will become clear later, let's number the steps in decreasing order so that the last step, when there are no oranges left, is zero. Then, the recurrence relation for the number of oranges in one step, *i*-1, given the number of oranges in the previous step, *i*, is as stated:

$$n_{i-1} = \frac{n_i}{2} - \frac{1}{2}$$

A little algebra gets the inverse recurrence relation, i.e., the number of oranges in one step given the number of oranges in the next step:

$$n_i = 2\,n_{i-1} + 1$$

Starting with zero oranges in step 0, we use the second relation to calculate the number of oranges in each previous step. See that, in the table below, an entry in the second column of one row plugs the entry from the third column of the preceding row into the expression on the right hand side of the equation. The third column results from carrying out the multiplication indicated by the second.

i	$2\,n_{i-1}+1$	n_i	2^i-1	binary
0		0	2^0-1	00000_2
1	$2(0)+1$	1	2^1-1	00001_2
2	$2(1)+1$	2^1+1	2^2-1	00011_2
3	$2(2^1+1)+1$	2^2+2^1+1	2^3-1	00111_2
4	$2(2^2+2^1+1)+1$	$2^3+2^2+2^1+1$	2^4-1	01111_2

The fifth column gives the binary representation of the third column, making it obvious that the fourth column calculates the number of oranges in a step directly (without recursion). The table is continued to the fourth step to reinforce the pattern, though the answer to the original problem is in row three.

The binary pattern further points out that the recurrence function is equivalent to a binary left shift followed by an increment.

Hank Dolben 2007 June 21